Last time

We obtained the T-matrix for scattering of two particles in vaccuum in two different ways

(1) Solving Schrodinger equation with a pseudo-potential to model inter-atomic interactions

Scattering Energy

$$T(E) = \frac{4\pi/m}{\frac{1}{a} + i\sqrt{mE/\hbar^2}} \qquad \mu = m/2$$

Scattering length

(2) Using Green function and a pt. contact interaction

$$\frac{1}{V_0} = \frac{1}{T(E)} - \int \frac{dk}{2\epsilon_k - E} \qquad \epsilon_k = k^2/2m$$

strength of pt. contact interaction

At this point we made a note that when we do many-body calculations and we see $\frac{1}{V_0}$ we should replace it with the T-matrix:

$$\frac{1}{V_0} \to \frac{1}{T(E=0)} - \int \frac{dk}{2\epsilon_k} = \frac{ma}{4\pi} - \int \frac{dk}{2\epsilon_k}$$

Comparing this expression with the gap equation of BCS theory

$$-\frac{1}{V_0} = \int \frac{dk}{2\sqrt{\Delta^2 + \xi_k^2}}$$

We observed that the integrals in the two were divergent in the same way.

$$\int \frac{dk}{2\epsilon_k} = \frac{1}{(2\pi)^3}\int_0^\infty k^2\,dk \int_0^\pi \sin\theta\, d\theta \int_0^{2\pi} d\phi \;\frac{m}{k^2} = \frac{4\pi}{(2\pi)^3}\int_0^\infty \frac{m\cancel{k^2}\,dk}{\cancel{k^2}} = \frac{m}{2\pi^2}\int_0^\infty dk$$

$$\int \frac{dk}{2\sqrt{\Delta^2+\xi_k^2}} = \frac{1}{2\pi^2}\int \frac{k^2\,dk}{2\sqrt{\Delta^2 + \left(\frac{k^2}{2m} - \mu\right)^2}}$$

$$\lim_{k\to\infty} 2\sqrt{\Delta^2 + \epsilon_k^2 - 2\mu\epsilon_k + \mu^2} = 2\epsilon_k\sqrt{1 - \frac{2\mu}{\epsilon_k} + \frac{\Delta^2+\mu^2}{\epsilon_k^2}} = 2\epsilon_k\left[1 - \frac{\mu}{\epsilon_k} + O\left(\frac{1}{\epsilon_k^2}\right)\right]$$

$$= \frac{1}{2\pi^2}\left[\int_0^\Lambda \frac{k^2\,dk}{2\sqrt{\Delta^2+\xi_k^2}} + \int_\Lambda^\infty \frac{k^2\,dk}{2\epsilon_k(1-\mu/\epsilon_k)}\right] = \frac{1}{2\pi^2}\left[\int_0^\Lambda \frac{k^2\,dk}{2\sqrt{\Delta^2+\xi_k^2}} + \int_\Lambda^\infty \frac{k^2\,dk}{2\epsilon_k}\left(1 + \frac{\mu}{\epsilon_k}\right)\right]$$

Λ is chosen to be sufficiently large to make $\epsilon_k \gg \mu, \Delta$

concentrating on the last term we find:

$$\frac{1}{2\pi^2}\int_\Lambda^\infty \frac{k^2\,dk}{k^2/m}\left[1+\frac{2m\mu}{k^2}\right] = \frac{1}{\pi^2}\int_\Lambda^\infty m\,dk + \frac{1}{2\pi^2}\int_\Lambda^\infty \frac{2m^2\mu}{k^2}\,dk$$

0 as $\Lambda\to\infty$

$$= \frac{1}{2\pi^2}\int_\Lambda^\infty m\,dk - \frac{m^2\mu}{\pi^2}\left(\frac{1}{k}\right)_\Lambda^\infty = \underbrace{\frac{m}{2\pi^2}\int_\Lambda^\infty dk}_{\text{exactly same form as integral in T-matrix}} + \cancel{\frac{m^2\mu}{\pi^2\Lambda}}$$

⇒ Putting these notions together we find that the gap equation is

$$-\frac{m}{4\pi a} = \int dk\left[\frac{1}{2\sqrt{\Delta^2+\xi_k^2}} - \frac{1}{2\epsilon_k}\right] \qquad (***)$$

where the integral on the RHS is completly convergent!

⇒ This means that divergence in the gap equation was caused Not by many-body physics but by 2-body physics. Getting the 2-body physics correct using the T-matrix from the exact solution of the 2-particle schrodinger equation fixed this "spurious" divergence.

⇒ Fixing divergences of this sort is called "regularization"

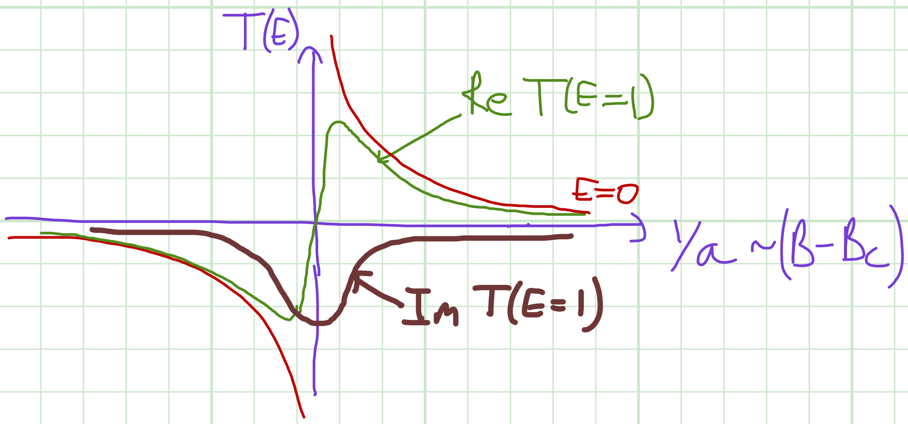
⇒ In $(***)$ it is useful to parametrize interaction strength by the dimensionless parameter

$$\frac{1}{k_F a}$$

where a is the scattering length and k_F is the Fermi momentum of the non-interacting gas $k_F = (6\pi^2 n)^{1/3}\hbar$.

A unified mean-field theory of the BCS – BEC crossover

Note: the T-matrix is singular on Resonance only for zero energy scattering. Let's plot $T(E)$ for finite E as a function of $1/a \sim B - B_{resonance}$



The BEC – BCS crossover – consider a gas of fermions near the Feshbach resonance. In our Fermi gas we have the intrinsic scale set by the Fermi energy. Most collisions occur at energy and momentum scales set by E_F and $k_F = \sqrt{2mE_F} = (6\pi^2 n)^{1/3}\hbar$.

$$\frac{1}{(2\pi)^3}\frac{4}{3}\pi\frac{k_F^3}{\hbar^3} = n$$

$$\hbar(6\pi^2 n)^{1/3} = k_F$$

Hence for typical collisions the T-matrix has no singularities even on resonance. As a result we should be able to use the T-matrix to describe the Fermi gas all the way across the resonance.

Indeed the gap equation ($\cancel{***}$) remains valid all the way across the resonance: there are no singularities at all for entire range: $\left[-\infty < \frac{1}{k_F a} < \infty\right]$.

BCS Side: $\frac{1}{k_F a} \ll -1$

Just as before, we can solve the BCS gap equation. There is only a slight modification of the resulting solution: the Debye scale get's replaced by the only other availble energy scale the Fermi Energy, resulting in the expression

$$\Delta = \hbar\omega_D e^{-\frac{1}{N(0)V}} \rightarrow \Delta = \frac{8E_F}{e^2} e^{-\frac{\pi}{2k_F|a|}}$$

e: 2.718...

$$E_F = \frac{(6\pi^2 n)^{2/3}\hbar^2}{2m}$$

n: density per spin

BEC side: $\frac{1}{k_F a} \gg 1$

Here we have a small problem ⇒ as the bound state develops the binding energy per fermion becomes large. In order to control the fermion density we must compensate by lowering the chemical potential μ. Thus we follow Leggett and supplement the gap equation with a particle # equation:

$$n = \sum_k v_k^2 = \int dk \, \frac{1}{2}\left(1 - \frac{\xi_k}{\sqrt{\Delta^2 + \xi_k^2}}\right)$$

particle # per spin
total # $= 2n$

$\xi_k = \epsilon_k - \mu$

Deep in the BEC phase

$$\mu \simeq -\frac{1}{2}E_b = -\frac{\hbar^2}{2ma^2} \qquad \Delta = \frac{4\,\epsilon_F}{\sqrt{3\pi\, k_F a}} \quad \Rightarrow \quad \Delta_{gap} = \sqrt{\Delta^2 + \mu^2} \sim |\mu| = \frac{\hbar^2}{2ma}$$

$k_F = \hbar\,(6\pi^2 n)^{1/3} \qquad \epsilon_F = k_F^2/2m$

are expressions for a non-interacting Fermi sea.

In the above Δ has the following meaning:

$$\Delta^2 \sim (\text{particle density in condensate}) \sim \frac{6\pi^2 k_F^3}{a}$$

needed to make units work

Unitary Regime:

When $|k_F a| \gg 1$, the exact value of a becomes irrelevant, and the only remaining scale in the problem is ϵ_F.
In this regime the gap and particle # equations have to be solved numerically yielding

$$\Delta = 0.68\,\epsilon_F, \qquad \mu = 0.59\,\epsilon_F$$

Critical temperature

We can extend our approach to the computation of T_c.

From BCS theory, we have

$$-\frac{1}{V_0} = \int d^3k \, \frac{1}{2\sqrt{\Delta^2+\xi_k^2}} \tanh\left(\frac{\sqrt{\Delta^2+\xi_k^2}}{2k_BT}\right)$$

setting $\Delta=0$ we obtain the T_c equation

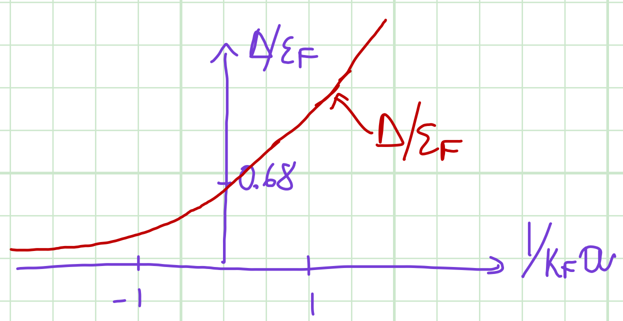
$$-\frac{1}{V_0} = \int \frac{d^3k}{2|\xi_k|} \tanh\left(\frac{|\xi_k|}{2k_BT_c}\right) = -\frac{m}{4\pi a} + \int \frac{dk}{2\varepsilon_k}$$

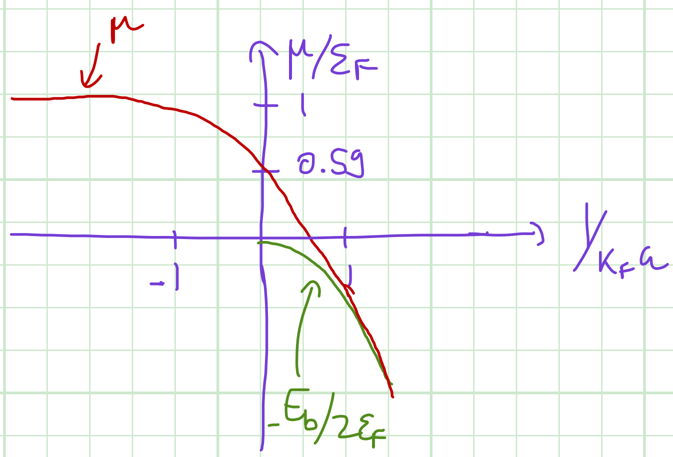
Solving this equation consistently with the atom # eq, we find

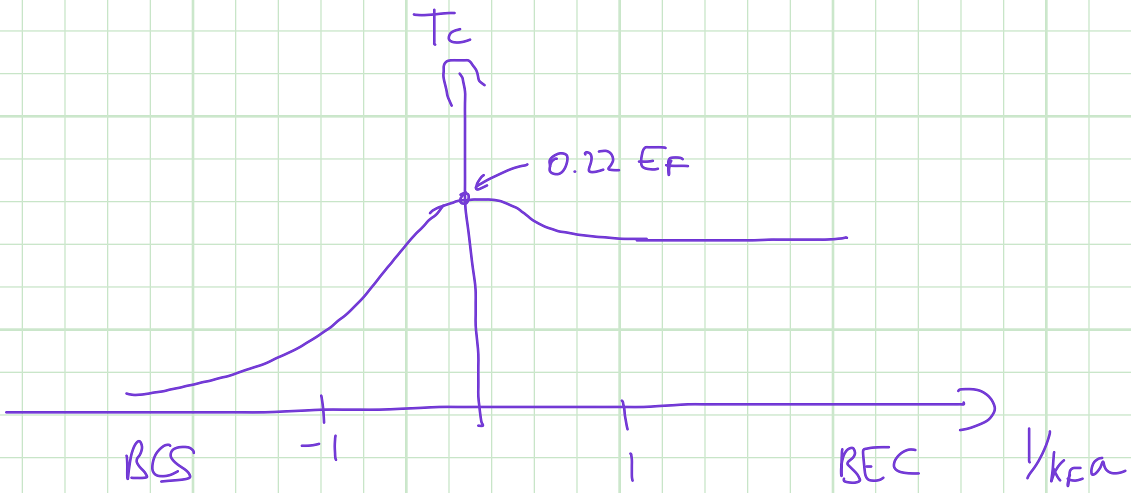
$$T_c \simeq \frac{8e^{\gamma_E}}{\pi e^2} E_F \, e^{-\frac{\pi}{2k_F|a|}} \qquad \text{BCS Regime} \qquad (\gamma_E = 0.57772\ldots)$$

$$T_c = 0.218\, \varepsilon_F \qquad \text{BEC Regime}$$

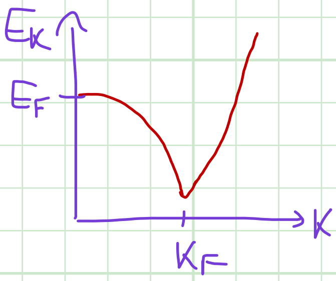
Behavior Across the resonance





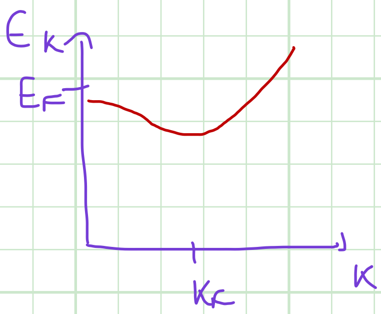


Quasi-particle dispersion across the resonance



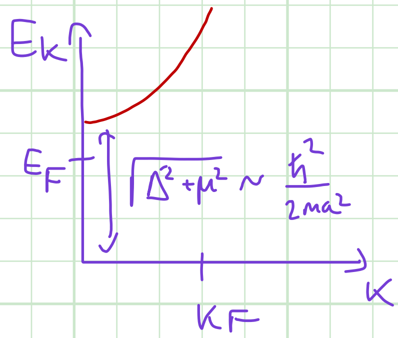
$1/k_F a = -2$

(deep BCS regime)



$1/k_F a = 0$

(unitary regime)



$1/k_F a = 2$

(deep BEC regime)

More involved theories

⇒ Fluctuations of order parameter — Diener, Sensarma, Randeria '08

⇒ Additional corrections via more involved trial wave functions — Haussmann, Rantner, Cerrito, Zwerger '07

⇒ Quantum Monte Carlo

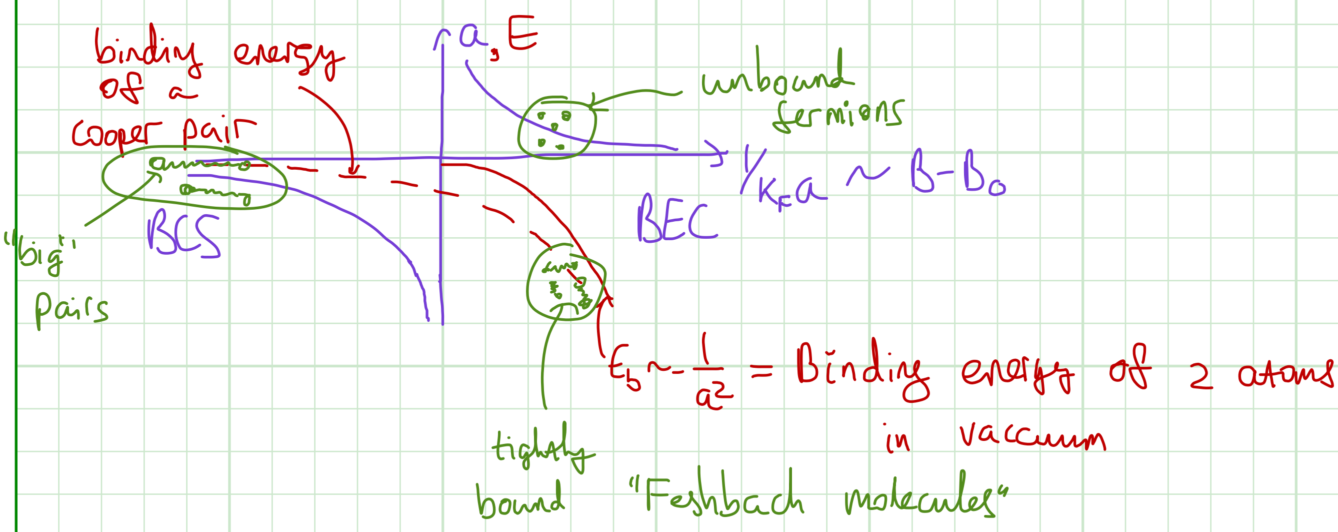
- Diagrammatic — Burovski, Kozik, Prokof'ev, Svistunov, Troyer '08

Van Houcke et al '11

Improved estimates of T_c, $\Delta(T)$, and thermodynamic observables like pressure, entropy, heat capacity, etc.

Experimental testing

First generation: projection experiments



Goal : see superfluidity in BCS and unitary regime

Problem: How to detect superfluidity ⇒

⇒ Cannot use TOF as individual fermions not phase coherent with each other, coherence is encoded in Cooper pairs

Solution: project Cooper pairs onto tightly bound molecules and free fermions deep on the BEC side using rapid B-sweep. interference between the molecules in TOF experiment related to coherence between initial Cooper pairs

Following quench of B-field

Molecule created by operator $b_q = \int dk \; \phi(k) \; c^+_{\frac{q}{2}+k\uparrow} \, c^+_{\frac{q}{2}-k\downarrow}$

U_k/u_k ⇒ wave function of the molecule

Schematically # molecules made is during quench is given by:

$(\#\text{ of molecules})_q = \langle \Psi_{BCS} | b_q^+ b_q | \Psi_{BCS} \rangle$

wave function before quench

$$= \delta(q) \left| \int dk \; \phi(k) \langle c^+_{k\uparrow} c^+_{-k\downarrow} \rangle \right|^2 + \int dk \; |\phi_k|^2 \langle n_{\frac{q}{2}+k\uparrow} \rangle \langle n_{\frac{q}{2}-k\downarrow} \rangle$$

"coherent" peak ⇒ condensed atoms → molecules

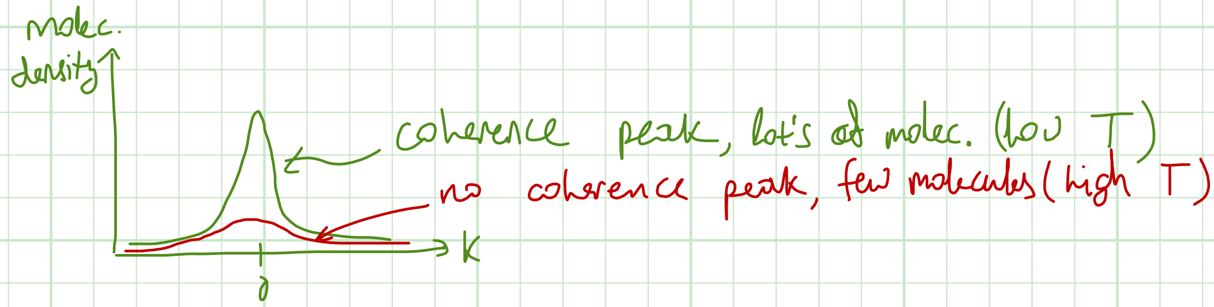
broad peak ⇒ non-condensed atoms → molecules

Less schematically, B-field cannot be changed instantaneously and details of the quench are important. This makes interpretation of data complicated.

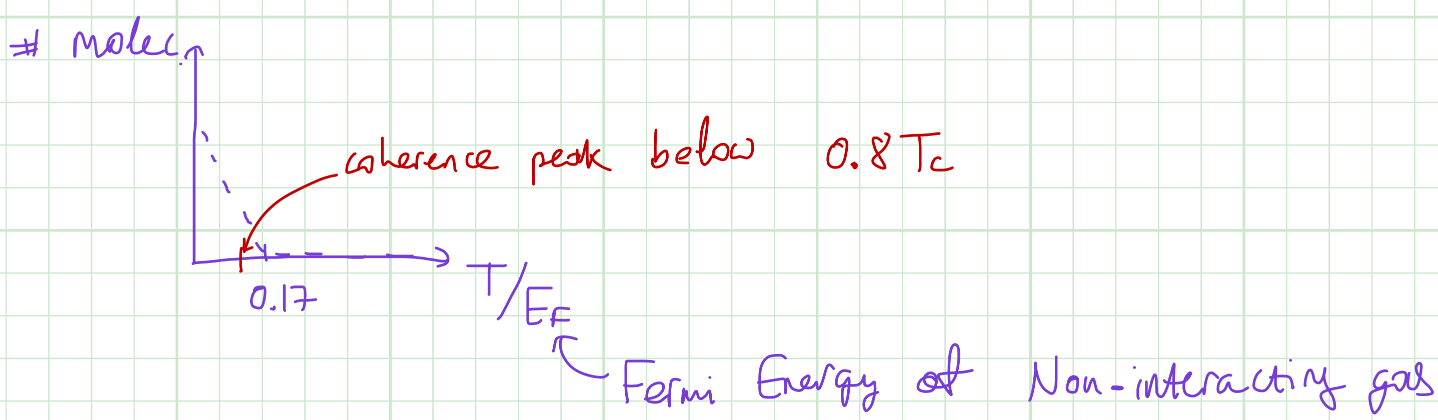
First observation of pairing in unitary fermions:

Greiner et. al Nature 426, 537 (2003)

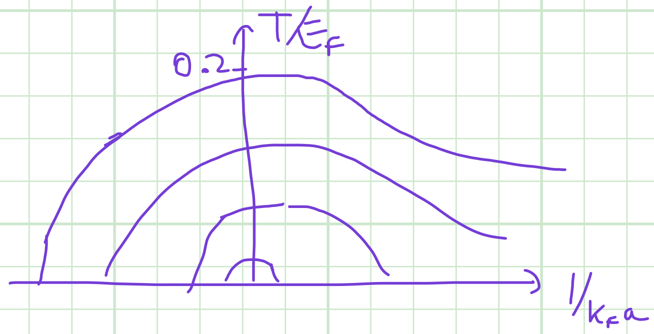
$\Rightarrow$ observe:



By mesuring # and momentum distribution of resulting molecules the manuscript aims to estimate the # and phase coherence of the original pairs.



Subsequent experiments map out the crossover all the way from weak BCS superfluidity to strong BEC regime.



Regal, Greiner, Jin PRL '04 $\Rightarrow$ ^{40}K

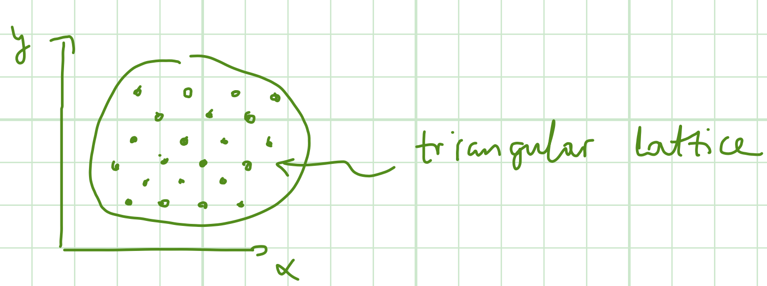
Zwierlein, et al, PRL '04 $\rightarrow$ ^{6}Li

Demonstration of coherence: See Ketterle + Zwierlein arXiv: 0801.2500

(1) make vortex lattice in BCS by spinning gas [with 2 stirring beams]

(2) Project onto BEC

(3) observe vortex lattice



$\Rightarrow$ measure critical temperature + polarization by detecting at what temperature + polarization vortices disappear

Noise correlations   See Greiner, Regal, Stewart, Jin PRL '05

$\Rightarrow$ Prepare BCS gas

$\Rightarrow$ Quench B-field to weakly interacting regime & remove trap

↳ Cooper pairs mapped onto single particle states

$\Rightarrow$ Perform TOF imaging

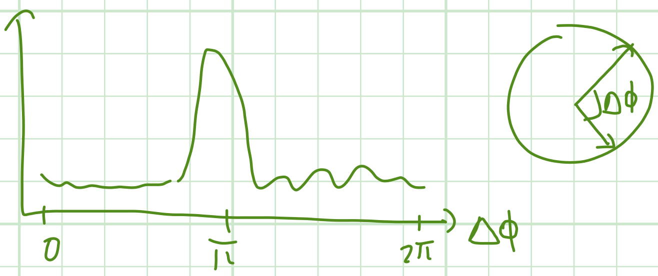
$\Rightarrow$ For each k $\Rightarrow$ $\Psi_k = (u_k + v_k c^+_{k\uparrow} c^+_{-k\downarrow})|0\rangle$

if $|k\uparrow\rangle$ is occupied so is $|-k\downarrow\rangle$

if $|k\uparrow\rangle$ is empty so is $|-k\downarrow\rangle$

$\Rightarrow$ measure the correlator $\langle n_{k\uparrow} n_{-k\downarrow}\rangle$

Correlation



RF spectroscopy

High precision measurments